Inverses: From the Basics

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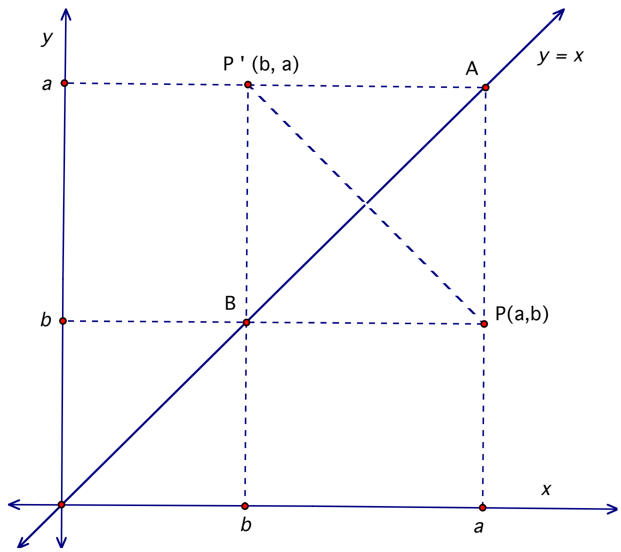
In an ordered pair, (a,b), the first item, a, determines the second item, b, through some rule. a is the ***input*** and b is the ***output***. The ordered pair changes, or ***maps***, a to b.

To undo that change, we need the ordered pair (b,a), which is called the ***inverse*** of (a,b). Because the multiplicative inverse of any real number x is $1/x = x^{-1}$, we use the -1 superscript to indicate an inverse; so $(a,b)^{-1} = (b,a)$ and $(b,a)^{-1} = (a,b)$. The -1 superscript here is NOT an exponent, and $(a,b)^{-1}$ is read "the inverse of ordered pair, or point, (a,b)".

If graphed on a coordinate grid, (a,b) and (b,a) are reflections of each other in the line $y = x$.



Proof: APBP′ is a square because all side lengths are $a - b$ and all angles are right. The diagonals of a square are perpendicular bisectors of each other, so P and P′ are reflections of each other.

This means that the inverse of any *set* (collection) of points is the reflection of that set in the line $y = x$.

Any set of ordered pairs is called a ***relation***, e.g., $S = \{(1,3),(2,4),(2,5)\}$. The inverse of S is $S^{-1} = \{(3,1),(4,2),(5,2)\}$. The ***domains*** and ***ranges*** of S and S^{-1} are shown in this ***mapping***:

S		S^{-1}	
Domain (D)	Range (R)	Domain (R)	Range (D)
1 →	3	3 →	1
2 → (to 4 and 5)	4	4 →	2
	5	5 → (to 2)	

Because one undoes what the other does, $S^{-1}(S(D)) = D$ and $S(S^{-1}(R)) = R$.

A ***function*** is a relation where each domain value maps to only one range value (hence the ***vertical line test*** to determine if a graph is a function). Above, S^{-1} is a function but S is not.

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A larger example: Let $T=\{(x,y)|\, y\ge x\}$, where the vertical bar means *such that*, so *T* is the set of all points above or in the line $y=x$. Then $T^{-1}=\{(x,y)|\, y\le x\}$, the reflection of *T* in the line $y=x$. Here neither *T* nor T^{-1} are functions.

Examples of using algebraic rules to define relations:

$c=\{(x,y)|\, x^2+y^2=25\}$, a circle centered at the origin with radius 5 (NOT a function);

$p(x)=\{(x,y)|\, y=3x^2+2\}$, a parabola (a function of *x*); and

$l(x)=\{(x,y)|\, y=3x-2\}$, a line (a function of *x*).

Set notation is usually dropped, with the understanding that a set of points is implied. So the above become: $c(x)=\pm\sqrt{25-x^2}$, $p(x)=3x^2+2$, and $l(x)=3x-2$.

To find the inverses of relations expressed with algebraic rules, let's use $l(x)=\{(x,y)|\, y=3x-2\}$ as an example. If we write it this way, $l=\{(x,3x-2)\}$, we can easily exchange the first and second components of each point to get $l^{-1}=\{(3x-2,x)\}$. But we want the second component to be the *y*-coordinate, so we can change the dummy variable *x* to *y* to get $l^{-1}=\{(3y-2,y)\}$. Then $x=3y-2$ and solving for *y*, we can restate the inverse as $l^{-1}(x)=\left\{(x,y)\,\middle|\, y=\frac{x+2}{3}\right\}$.

The same procedure in a streamlined fashion: $l(x)=3x-2=y$, exchange the variables, $3y-2=x$, solve for *y*, $y=\frac{x+2}{3}=l^{-1}(x)$.

Applying this procedure to the circle and parabola above we get: $c^{-1}=\{(x,y)|\, y^2+x^2=25\}=c$, it is its own inverse; and $p^{-1}(x)=\pm\sqrt{\frac{x-2}{3}}$, not a function.

Test for inverses: For two relations to be inverses of each other, each must undo what the other does; i.e., if $f(g(x))=x$ and $g(f(x))=x$ then $f=g^{-1}$ and $g=f^{-1}$.

Checking *l* and its proposed inverse, above: $l\left(l^{-1}(x)\right)=l\left(\frac{x+2}{3}\right)=3\left(\frac{x+2}{3}\right)-2=x$ and

$l^{-1}\left(l(x)\right)=l^{-1}(3x-2)=\frac{(3x-2)+2}{3}=x$, so they are inverses of each other.

To check whether an inverse is a function without graphing it, use the ***horizontal line test*** on the original relation's graph. If any horizontal line intersects the original relation's graph in more than one point, the inverse is not a function. This works because the inverse of a horizontal line is a vertical line, so you're effectively doing a vertical line test on the inverse.

Functions whose inverses are functions are called ***one-to-one*** functions.