

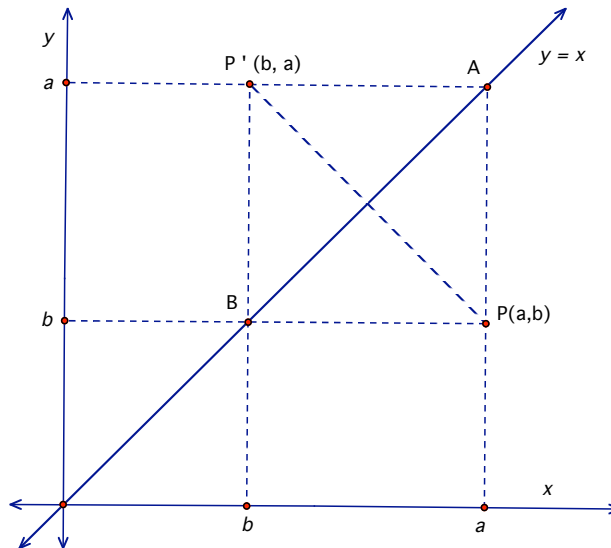
## Inverses: From the Basics

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In an ordered pair,  $(a,b)$ , the first item,  $a$ , determines the second item,  $b$ , through some rule.  $a$  is the **input** and  $b$  is the **output**. The ordered pair changes, or **maps**,  $a$  to  $b$ .

To undo that change, we need the ordered pair  $(b,a)$ , which is called the **inverse** of  $(a,b)$ . Because the multiplicative inverse of any real number  $x$  is  $1/x = x^{-1}$ , we use the  $-1$  superscript to indicate an inverse; so  $(a,b)^{-1} = (b,a)$  and  $(b,a)^{-1} = (a,b)$ . The  $-1$  superscript here is NOT an exponent, and  $(a,b)^{-1}$  is read “the inverse of ordered pair, or point,  $(a,b)$ ”.

If graphed on a coordinate grid,  $(a,b)$  and  $(b,a)$  are reflections of each other in the line  $y = x$ .



Proof:  $APBP'$  is a square because all side lengths are  $a - b$  and all angles are right. The diagonals of a square are perpendicular bisectors of each other, so  $P$  and  $P'$  are reflections of each other.

**This means that the inverse of any set (collection) of points is the reflection of that set in the line  $y = x$ .**

Any set of ordered pairs is called a **relation**, e.g.,  $S = \{(1,3),(2,4),(2,5)\}$ . The inverse of  $S$  is  $S^{-1} = \{(3,1),(4,2),(5,2)\}$ . The **domains** and **ranges** of  $S$  and  $S^{-1}$  are shown in this **mapping**:

S		$S^{-1}$	
Domain (D)	Range (R)	Domain (R)	Range (D)
1	→ 3	3	→ 1
2	→ 4	4	→ 2
	→ 5	5	→ 2

Because one undoes what the other does,  $S^{-1}(S(D)) = D$  and  $S(S^{-1}(R)) = R$ .

A **function** is a relation where each domain value maps to only one range value (hence the **vertical line test** to determine if a graph is a function). Above,  $S^{-1}$  is a function but  $S$  is not.

A larger example: Let  $T = \{(x, y) \mid y \geq x\}$ , where the vertical bar means *such that*, so  $T$  is the set of all points above or in the line  $y = x$ . Then  $T^{-1} = \{(x, y) \mid y \leq x\}$ , the reflection of  $T$  in the line  $y = x$ . Here neither  $T$  nor  $T^{-1}$  is a function.

Examples of using algebraic rules to define relations:

$c = \{(x, y) \mid x^2 + y^2 = 25\}$ , a circle centered at the origin with radius 5 (NOT a function);

$p(x) = \{(x, y) \mid y = 3x^2 + 2\}$ , a parabola (a function of  $x$ ); and

$l(x) = \{(x, y) \mid y = 3x - 2\}$ , a line (a function of  $x$ ).

Set notation is usually dropped, with the understanding that a set of points is implied. So the above become:  $c(x) = \pm\sqrt{25 - x^2}$ ,  $p(x) = 3x^2 + 2$ , and  $l(x) = 3x - 2$ .

**To find the inverses of relations expressed with algebraic rules**, let's use

$l(x) = \{(x, y) \mid y = 3x - 2\}$  as an example. If we write it this way,  $l = \{(x, 3x - 2)\}$ , we can easily exchange the first and second components of each point to get  $l^{-1} = \{(3x - 2, x)\}$ . But we want the second component to be the  $y$ -coordinate, so we can change the dummy variable  $x$  to  $y$  to get  $l^{-1} = \{(3y - 2, y)\}$ . Then  $x = 3y - 2$  and solving for  $y$ , we can restate the inverse as

$$l^{-1}(x) = \left\{ (x, y) \mid y = \frac{x + 2}{3} \right\}.$$

The same procedure in a streamlined fashion:  $l(x) = 3x - 2 = y$ , exchange the variables,

$$3y - 2 = x, \text{ solve for } y, y = \frac{x + 2}{3} = l^{-1}(x).$$

Applying this procedure to the circle and parabola above we get:  $c^{-1} = \{(x, y) \mid y^2 + x^2 = 25\} = c$ , it is its own inverse; and  $p^{-1}(x) = \pm\sqrt{\frac{x - 2}{3}}$ , not a function.

**Test for inverses:** For two relations to be inverses of each other, each must undo what the other does; i.e., if  $f(g(x)) = x$  and  $g(f(x)) = x$  then  $f = g^{-1}$  and  $g = f^{-1}$ .

Checking  $l$  and its proposed inverse, above:  $l(l^{-1}(x)) = l\left(\frac{x + 2}{3}\right) = 3\left(\frac{x + 2}{3}\right) - 2 = x$  and

$$l^{-1}(l(x)) = l^{-1}(3x - 2) = \frac{(3x - 2) + 2}{3} = x, \text{ so they are inverses of each other.}$$

**To check whether an inverse is a function without graphing it**, use the *horizontal line test* on the original relation's graph. If any horizontal line intersects the original relation's graph in more than one point, the inverse is not a function. This works because the inverse of a horizontal line is a vertical line, so you're effectively doing a vertical line test on the inverse.

Functions whose inverses are functions are called *one-to-one* functions.