Graphing Rational Functions

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To graph rational function $f(x)=\dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials:

1. Find the zeros of $D(x)$, as they are NOT in the domain of $f(x)$. State the domain of $f(x)$.
2. Find the zeros of $N(x)$. For each and every zero of $N(x)$ that is also a zero of $D(x)$, say $x=a$, there is an $\dfrac{x-a}{x-a}$ factor in $f(x)$ that has a value one if $x \neq a$, and is undefined if $x=a$. This causes a "hole" in $f(x)$ at $x=a$. Eliminate such common factors in $N(x)$ and $D(x)$. Now $f(x)=\dfrac{n(x)}{d(x)}$ with no common factors in $n(x)$ and $d(x)$, but the domain of $f(x)$ is still that stated in (1), with some holes possibly identified. A hole at $x=a$ has a y-coordinate $=\dfrac{n(a)}{d(a)}$.
3. The zeros of $f(x)$ are those of $n(x)$ and, if real, its x-intercepts. Its y-intercept $= f(0)$, if it exists.
4. If the degree of $n(x) \geq$ degree of $d(x)$, divide $n(x)$ by $d(x)$, using long division. Whether you divided or not, you should now have a function of the form $f(x)=q(x)+\dfrac{r(x)}{d(x)}$. Note that the degree of $r(x)$ is less than the degree of $d(x)$, and $q(x)=0$ if you didn't divide.
5. Whenever $d(x)\to 0$, $f(x)\to\infty$, so the zeros of $d(x)$ are the vertical asymptotes of $f(x)$. Determine the behavior on either side of each vertical asymptote; i.e.: $+\infty$ or $-\infty$.
6. When $x\to\infty$, $f(x)\to q(x)$ because $\dfrac{r(x)}{d(x)}\to\dfrac{1}{x^k}\to 0$, so $y=q(x)$ is the non-vertical asymptote of $f(x)$. Determine whether $f(x)$ is slightly above or below $q(x)$ at $+\infty$ or $-\infty$.
7. Sketch what you know so far (asymptotes and behaviors near them, holes and zeros of $f(x)$, and $f(0)$); then find more points that will help you shape the graph.
8. Complete the graph with smooth curved lines.