

## The Remainder Theorem and Synthetic Substitution/Division

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Let  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$  be a polynomial of degree  $n$ .

If  $P(x)$  is divided by  $x - b$ , then

$$\frac{P(x)}{x - b} = Q(x) + \frac{R}{x - b},$$

where  $Q(x)$  is the quotient, a polynomial of degree  $n - 1$ , and  $R$  is the remainder, a constant. Multiplying both sides by  $x - b$  gives

$$P(x) = (x - b)Q(x) + R.$$

If the polynomial is evaluated at  $x = b$ , then

$$P(b) = (b - b)Q(b) + R = 0 \cdot Q(b) + R = R. \text{ So } R = P(b) \text{ and}$$

$$P(x) = (x - b)Q(x) + P(b). \text{ This is } \mathbf{The\ Remainder\ Theorem}.$$

**So, to find  $P(b)$  you can divide  $P(x)$  by  $x - b$  and the remainder will be  $P(b)$ .**

Also,  **$P(b) = 0$  if and only if  $x - b$  is a factor of  $P(x)$** ; this is ***The Factor Theorem***.

Of course you can always directly substitute  $b$  for  $x$  in the polynomial, like this:

$$P(b) = a_n b^n + a_{n-1} b^{n-1} + \dots + a_1 b + a_0, \text{ and using some unusual factoring,}$$

$$P(b) = (((a_n)b + a_{n-1})b + \dots + a_1)b + a_0, \text{ which is called the } \mathbf{nested\ form}.$$

The ***nested form*** represents a procedure that can be tabularized:

	$x^n$	$x^{n-1}$	$\dots$	$x^1$	$1$
$b$	$a_n$	$a_{n-1}$	$\dots$	$a_1$	$a_0$
(+)	$(a_n)b$	$\dots$	$(((a_n)b + a_{n-1})b + \dots + a_2)b$	$(((a_n)b + a_{n-1})b + \dots + a_2)b + a_1$	$(((a_n)b + a_{n-1})b + \dots + a_2)b + a_1$
	$a_n$	$(a_n)b + a_{n-1}$	$\dots$	$(((a_n)b + a_{n-1})b + \dots + a_2)b + a_1$	$P(b)$

This is known as ***Synthetic Substitution***. For a concrete example, let

$f(x) = 2x^3 - 5x^2 + 7$ , then  $f(3)$  is found using ***Synthetic Substitution*** as follows in the figure at the left.

	$x^3$	$x^2$	$x^1$	$1$	
$3$	$2$	$-5$	$0$	$7$	
(+)	$6$	$3$	$9$	$9$	
	$2$	$1$	$3$	$16$	$= f(3)$

  

	$x^3$	$x^2$	$x^1$	$1$	
$3$	$2$	$-5$	$0$	$7$	$\longleftarrow f(x)$
(+)	$6$	$3$	$9$	$9$	$\longleftarrow bQ(x)$
	$2$	$1$	$3$	$16$	$= f(3)$

The Remainder Theorem can be used to understand what else is being shown in ***Synthetic Substitution***. If  $P(x) = (x - b)Q(x) + P(b)$ , then, with a little algebraic manipulation,  $P(x) + bQ(x) = xQ(x) + P(b)$ ; which shows that ***Synthetic Substitution*** also finds the quotient  $Q(x)$  of a division of  $P(x)$  by  $x - b$ , as annotated for our concrete example in the above figure on the right. So ***Synthetic Substitution*** is also known as ***Synthetic Division***.

Historically Nested Form and Synthetic Substitution were used to evaluate polynomials before calculators and are used in computer programs today. They can also be used effectively with just a four function calculator.