The Remainder Theorem and Synthetic Substitution/Division

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Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ be a polynomial of degree n.

If $P(x)$ is divided by $x - b$, then

$$\frac{P(x)}{x-b} = Q(x) + \frac{R}{x-b},$$

where $Q(x)$ is the quotient, a polynomial of degree $n - 1$, and R is the remainder, a constant. Multiplying both sides by $x - b$ gives

$$P(x) = (x-b)Q(x) + R.$$

If the polynomial is evaluated at $x = b$, then

$P(b) = (b-b)Q(b) + R = 0 \cdot Q(b) + R = R$. So $R = P(b)$ and

$P(x) = (x-b)Q(x) + P(b)$. This is ***The Remainder Theorem***.

So, to find $P(b)$ you can divide $P(x)$ by $x - b$ and the remainder will be $P(b)$.

Also, **$P(b) = 0$ if and only if $x - b$ is a *factor* of $P(x)$**; this is ***The Factor Theorem.***

Of course you can always directly substitute b for x in the polynomial, like this:

$P(b) = a_n b^n + a_{n-1}b^{n-1} + ... + a_1 b + a_0$, and using some unusual factoring,

$P(b) = (((a_n)b + a_{n-1})b + ... + a_1)b + a_0$, which is called the ***nested form***.

The ***nested form*** represents a procedure that can be tabularized:

$$\begin{array}{r|llllc}
 & x^n & x^{n-1} & \cdots & x^1 & 1 \\
b & a_n & a_{n-1} & \cdots & a_1 & a_0 \\
(+) & & (a_n)b & \cdots & (((a_n)b+a_{n-1})b+\ldots+a_2)b & ((((a_n)b+a_{n-1})b+\ldots+a_2)b+a_1)b \\
\hline
 & a_n & (a_n)b+a_{n-1} & \cdots & (((a_n)b+a_{n-1})b+\ldots+a_2)b+a_1 & \boxed{P(b)}
\end{array}$$

This is known as ***Synthetic Substitution***. For a concrete example, let $f(x) = 2x^3 - 5x^2 + 7$, then $f(3)$ is found using ***Synthetic Substitution*** as follows in the figure at the left.

$$\begin{array}{r|rrrr}
 & x^3 & x^2 & x^1 & 1 \\
3 & 2 & -5 & 0 & 7 \\
(+) & & 6 & 3 & 9 \\
\hline
 & 2 & 1 & 3 & \boxed{16}
\end{array} = f(3)
\qquad
\begin{array}{rr|rrrrl}
 & & x^3 & x^2 & x^1 & 1 & \\
 & 3 & 2 & -5 & 0 & 7 & \longleftarrow f(x) \\
 & (+) & & 6 & 3 & 9 & \longleftarrow bQ(x) \\
\hline
xQ(x) \longrightarrow & & 2 & 1 & 3 & \boxed{16} & = f(3)
\end{array}$$

The Remainder Theorem can be used to understand what else is being shown in ***Synthetic Substitution***. If $P(x) = (x-b)Q(x) + P(b)$, then, with a little algebraic manipulation, $P(x) + bQ(x) = xQ(x) + P(b)$; which shows that ***Synthetic Substitution*** also finds the quotient $Q(x)$ of a division of $P(x)$ by $x - b$, as annotated for our concrete example in the above figure on the right. So ***Synthetic Substitution*** is also known as ***Synthetic Division***.

Historically Nested Form and Synthetic Substitution were used to evaluate polynomials before calculators and are used in computer programs today. They can also be used effectively with just a four function calculator.