Derivation of Derivatives: *Functions*

Polynomial Functions:

If $n > 0$ is an integer and $y = x^n$, then $\dfrac{dy}{dx} = \lim\limits_{h\to 0}\dfrac{(x+h)^n - x^n}{h}$

$$= \lim_{h\to 0}\frac{-x^n + \binom{n}{n}x^n h^0 + \binom{n}{n-1}x^{n-1}h^1 + \binom{n}{n-2}x^{n-2}h^2 + \cdots + \binom{n}{0}x^0 h^n}{h}$$

$$= \lim_{h\to 0}\frac{-x^n + x^n + \binom{n}{n-1}x^{n-1}h^1 + \binom{n}{n-2}x^{n-2}h^2 + \cdots + h^n}{h}$$

$$= \lim_{h\to 0}\left[\binom{n}{n-1}x^{n-1} + \binom{n}{n-2}x^{n-2}h^1 + \cdots + h^{n-1}\right] \boxed{\Rightarrow \frac{d}{dx}\left(x^n\right) = nx^{n-1}} \qquad (1)$$

Use the product rule for more complicated monomials, and the sum rule for more than one term.

Inverse Functions:

If $y = f(x)$ and $\dfrac{dy}{dx} = \dfrac{d}{dx}\big(f(x)\big)$, then $f^{-1}(y) = x$ and $\dfrac{d}{dx}\big(f^{-1}(y)\big)\cdot\dfrac{dy}{dx} = 1$, by chain rule.

$$\Rightarrow \frac{d}{dx}\left(f^{-1}(y)\right) = \frac{dx}{dy} = \left[\frac{d}{dx}\big(f(x)\big)\right]^{-1} \boxed{\Rightarrow \text{Inverses have reciprocal slopes at inverse points.}} \qquad (2)$$

Natural Log Function:

Because the Natural Log Function y=ln x is defined as the area under the curve y=1/t from 1 to x, the derivative of ln x is 1/x. Using the chain rule, the derivative of ln u = 1/u * du/dx.

Natural Exponential Function:

The equivalent logarithmic function to y = e^x is x = ln y. Taking the implicite derivative of both sides of the latter gives 1 = 1/y*dy/dx, so dy/dx = y; or d(e^x)/dx = e^x. e^x is its own derivative.

y= b^x = e^(x ln b), so by chain rule, y' = (ln b)e^(x ln b)= (ln b)b^x

Any Power of x Function:

If y=x^n where n is any real, then y=(e^(ln x))^n=e^(n ln x).

dy/dx = (e^(n ln x))*(n/x) = (n/x)*x^n = n x^(n-1).