

Derivation of Derivatives: Rules

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Rule for Sum of Functions:

$$\begin{aligned} \text{If } f(x) = u(x) + v(x), \text{ then } \frac{d(f(x))}{dx} &= \lim_{h \rightarrow 0} \frac{u(x+h) + v(x+h) - u(x) - v(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} \Rightarrow \boxed{\frac{d}{dx}(u(x) + v(x)) = \frac{d(u(x))}{dx} + \frac{d(v(x))}{dx}} \end{aligned} \quad (1)$$

Rule for Product of Functions:

$$\begin{aligned} \text{If } f(x) = u(x) \cdot v(x), \text{ then } \frac{d(f(x))}{dx} &= \lim_{h \rightarrow 0} \frac{u(x+h) \cdot v(x+h) - u(x) \cdot v(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) \cdot v(x+h) - u(x) \cdot v(x+h) + u(x) \cdot v(x+h) - u(x) \cdot v(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} \cdot v(x+h) \right] + \lim_{h \rightarrow 0} \left[\frac{v(x+h) - v(x)}{h} \cdot u(x) \right] \\ &\Rightarrow \boxed{\frac{d}{dx}(u(x) \cdot v(x)) = \frac{d(u(x))}{dx} \cdot v(x) + \frac{d(v(x))}{dx} \cdot u(x)} \end{aligned} \quad (2)$$

Rule for Quotient of Functions:

$$\begin{aligned} \text{If } f(x) = \frac{u(x)}{v(x)}, \text{ then } \frac{d(f(x))}{dx} &= \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} = \lim_{h \rightarrow 0} \frac{u(x+h) \cdot v(x) - u(x) \cdot v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) \cdot v(x) - u(x) \cdot v(x) + u(x) \cdot v(x) - u(x) \cdot v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h \rightarrow 0} \left[\frac{1}{v(x+h)v(x)} \cdot \left[\frac{u(x+h) - u(x)}{h} \cdot v(x) - \frac{v(x+h) - v(x)}{h} \cdot u(x) \right] \right] \\ &\Rightarrow \boxed{\frac{d}{dx} \left(\frac{u(x)}{v(x)} \right) = \left[\frac{d(u(x))}{dx} \cdot v(x) - \frac{d(v(x))}{dx} \cdot u(x) \right] \cdot \frac{1}{v^2(x)}} \end{aligned} \quad (3)$$

Chain Rule for Composition of Functions:

$$\begin{aligned} \text{If } h(x) = f(g(x)), \text{ then } \frac{d(h(x))}{dx} &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ \text{Let } u = g(x) \text{ and } v = g(x+h) - g(x), \text{ then} \\ \frac{d(h(x))}{dx} &= \lim_{h \rightarrow 0} \left[\frac{f(u+v) - f(u)}{v} \cdot \frac{v}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{f(u+v) - f(u)}{v} \cdot \frac{g(x+h) - g(x)}{h} \right] \\ \text{As } h \rightarrow 0, v = g(x+h) - g(x) \rightarrow 0, \text{ so } \frac{d(h(x))}{dx} &= \lim_{v \rightarrow 0} \frac{f(u+v) - f(u)}{v} \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &\Rightarrow \boxed{\frac{d}{dx}(f(g(x))) = \frac{d(f(u))}{du} \cdot \frac{d(g(x))}{dx}} \end{aligned} \quad (4)$$

$$\boxed{\text{OR: If } y = f(u) \text{ and } u = g(x), \text{ then } \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx} \Rightarrow \text{just like fractions.}} \quad (5)$$