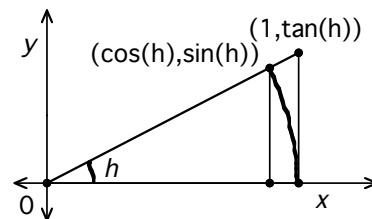


Derivation of Derivatives: Trigonometric Functions

© 2008-2008 Stephen K. Stephenson, sks23@cornell.edu. All Rights Reserved.

In the figure, small triangle area \leq sector area \leq large triangle area;



$$\Rightarrow \frac{1}{2} \cos(h) |\sin(h)| \leq \frac{|h|}{2\pi} \cdot \pi(1)^2 \leq \frac{1}{2} (1) |\tan(h)|;$$

$$\Rightarrow \cos(h) |\sin(h)| \leq |h| \leq \frac{|\sin(h)|}{\cos(h)}; \Rightarrow \cos(h) \leq \frac{h}{\sin(h)} \leq \frac{1}{\cos(h)}$$

$$\lim_{h \rightarrow 0} \cos(h) = 1; \Rightarrow 1 \leq \lim_{h \rightarrow 0} \frac{h}{\sin(h)} \leq 1; \Rightarrow \lim_{h \rightarrow 0} \frac{h}{\sin(h)} = 1; \Rightarrow \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1 \quad (1)$$

$$\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = \lim_{h \rightarrow 0} \left[\frac{\cos(h) - 1}{h} \cdot \frac{\cos(h) + 1}{\cos(h) + 1} \right] = \lim_{h \rightarrow 0} \frac{\cos^2(h) - 1}{h(\cos(h) + 1)}$$

$$= \lim_{h \rightarrow 0} \frac{-\sin^2(h)}{h(\cos(h) + 1)} = \lim_{h \rightarrow 0} \left[\frac{-\sin(h)}{h} \cdot \frac{\sin(h)}{\cos(h) + 1} \right] = -1 \cdot \frac{0}{1+1}; \Rightarrow \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0 \quad (2)$$

$$\frac{d(\sin(x))}{dx} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left[\sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h} \right] = \sin(x) \cdot 0 + \cos(x) \cdot 1 \Rightarrow \frac{d(\sin(x))}{dx} = \cos(x) \quad (3)$$

$$\frac{d(\cos(x))}{dx} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left[\cos(x) \cdot \frac{\cos(h) - 1}{h} - \sin(x) \cdot \frac{\sin(h)}{h} \right] = \cos(x) \cdot 0 - \sin(x) \cdot 1 \Rightarrow \frac{d(\cos(x))}{dx} = -\sin(x) \quad (4)$$

$$\frac{d(\tan x)}{dx} = \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\cos x \cos x - (-\sin x) \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \frac{d(\tan x)}{dx} = \sec^2 x \quad (5)$$

$$\frac{d(\cot x)}{dx} = \frac{d}{dx} \left(\frac{\cos x}{\sin x} \right) = \frac{(-\sin x) \sin x - \cos x \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x} \Rightarrow \frac{d(\cot x)}{dx} = -\csc^2 x \quad (6)$$

$$\frac{d(\sec x)}{dx} = \frac{d}{dx} \left(\frac{1}{\cos x} \right) = \frac{0 \cdot \cos x - (-\sin x) \cdot 1}{\cos^2 x} = \frac{\sin x}{\cos^2 x} \Rightarrow \frac{d(\sec x)}{dx} = \sec x \tan x \quad (7)$$

$$\frac{d(\csc x)}{dx} = \frac{d}{dx} \left(\frac{1}{\sin x} \right) = \frac{0 \cdot \sin x - \cos x \cdot 1}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} \Rightarrow \frac{d(\csc x)}{dx} = -\csc x \cot x \quad (8)$$

N.B.: (5) - (8) use quotient rule.