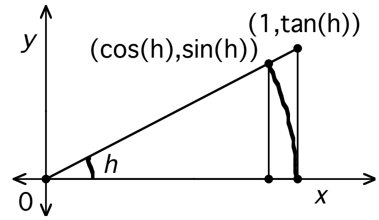
Derivation of Derivatives: ***Trigonometric Functions***



In the figure, small triangle area ≤ sector area ≤ large triangle area;

$$\Rightarrow \frac{1}{2}\cos(h)\left|\sin(h)\right| \le \frac{|h|}{2\pi}\cdot\pi(1)^2 \le \frac{1}{2}(1)\left|\tan(h)\right|;$$

$$\Rightarrow \cos(h)\left|\sin(h)\right| \le |h| \le \frac{\left|\sin(h)\right|}{\cos(h)};\Rightarrow \cos(h) \le \frac{h}{\sin(h)} \le \frac{1}{\cos(h)}$$

$$\lim_{h\to 0}\cos(h)=1;\Rightarrow 1 \le \lim_{h\to 0}\frac{h}{\sin(h)} \le 1;\Rightarrow \lim_{h\to 0}\frac{h}{\sin(h)}=1.\boxed{\Rightarrow \lim_{h\to 0}\frac{\sin(h)}{h}=1} \qquad (1)$$

$$\lim_{h\to 0}\frac{\cos(h)-1}{h}=\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\cdot\frac{\cos(h)+1}{\cos(h)+1}\right]=\lim_{h\to 0}\frac{\cos^2(h)-1}{h\left(\cos(h)+1\right)}$$

$$=\lim_{h\to 0}\frac{-\sin^2(h)}{h\left(\cos(h)+1\right)}=\lim_{h\to 0}\left[\frac{-\sin(h)}{h}\cdot\frac{\sin(h)}{\cos(h)+1}\right]=-1\cdot\frac{0}{1+1};\boxed{\Rightarrow \lim_{h\to 0}\frac{\cos(h)-1}{h}=0} \qquad (2)$$

$$\frac{d\left(\sin(x)\right)}{dx}=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}=\lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$=\lim_{h\to 0}\left[\sin(x)\cdot\frac{\cos(h)-1}{h}+\cos(x)\cdot\frac{\sin(h)}{h}\right]=\sin(x)\cdot 0+\cos(x)\cdot 1\boxed{\Rightarrow \frac{d\left(\sin(x)\right)}{dx}=\cos(x)} \qquad (3)$$

$$\frac{d\left(\cos(x)\right)}{dx}=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}=\lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

$$=\lim_{h\to 0}\left[\cos(x)\cdot\frac{\cos(h)-1}{h}-\sin(x)\cdot\frac{\sin(h)}{h}\right]=\cos(x)\cdot 0-\sin(x)\cdot 1\boxed{\Rightarrow \frac{d\left(\cos(x)\right)}{dx}=-\sin(x)} \qquad (4)$$

$$\frac{d(\tan x)}{dx}=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{\cos x\cos x-(-\sin x)\sin x}{\cos^2 x}=\frac{1}{\cos^2 x}\boxed{\Rightarrow \frac{d(\tan x)}{dx}=\sec^2 x} \qquad (5)$$

$$\frac{d(\cot x)}{dx}=\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)=\frac{(-\sin x)\sin x-\cos x\cos x}{\sin^2 x}=\frac{-1}{\sin^2 x}\boxed{\Rightarrow \frac{d(\cot x)}{dx}=-\csc^2 x} \qquad (6)$$

$$\frac{d(\sec x)}{dx}=\frac{d}{dx}\left(\frac{1}{\cos x}\right)=\frac{0\cdot\cos x-(-\sin x)\cdot 1}{\cos^2 x}=\frac{\sin x}{\cos^2 x}\boxed{\Rightarrow \frac{d(\sec x)}{dx}=\sec x\tan x} \qquad (7)$$

$$\frac{d(\csc x)}{dx}=\frac{d}{dx}\left(\frac{1}{\sin x}\right)=\frac{0\cdot\sin x-\cos x\cdot 1}{\sin^2 x}=\frac{-\cos x}{\sin^2 x}\boxed{\Rightarrow \frac{d(\csc x)}{dx}=-\csc x\cot x} \qquad (8)$$

N.B.: (5) - (8) use quotient rule.