Pebble Puzzle

An Activity Investigating

The Design Of Counting Board Abaci

Used By Ancient Peoples

And In The Modern Classroom

by

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For 4,000 years people did their arithmetic by placing and manipulating pebbles on a flat surface inscribed with lines. At first, the lines could have been furrows drawn with a stick in sand or dirt with the pebbles placed in or between them.

In classic Greek architecture, an abacus is a flat slab of marble on top of a column's capital, supporting the architrave, or beam. Such an abacus (perhaps chipped beyond use in construction) makes a fine flat surface on which to inscribe lines; from which we get the name, counting board abacus. Later, constrained bead devices with less arithmetic functionality were also called abaci, e.g., Roman Hand Abacus, Suan Pan, and Soroban.

From http://mathforum.org/library/drmath/view/57572.html:

> The Roman expression for "to calculate" is "calculus ponere" - literally, "to place pebbles". When a Roman wished to settle accounts with someone, he would use the expression "vocare aliquem ad calculos" - "to call them to the pebbles."

So the puzzle questions are:

1. If you were designing a counting board abacus, how would you draw the lines?

2. How many pebbles would you need to represent 9,834; or 9,834,000,000,000,000?

3. On your counting board, how many pebbles would you need to calculate 9,834 - 4,983 + 3,894, and how long would it take you?

Hint: The value of 9,834,000,000,000,000 can be represented with just 9 pebbles. How?

===== Answers to Pebble Puzzle =====

9834 = 9000 + 800 + 30 + 4

1st abacus design (your hands have 10 digits):

```
------------- 10000s

------------- 1000s

------------- 100s

------------- 10s

X------------ 1s     (X marks the unit line)
```

9834 with 24 pebbles:

```
------------- 10000s

--000000000-- 1000s

--00000000--- 100s

--000-------- 10s

X-0000------- 1s
```

(N.B.: Sorted aquarium pebbles work well and have a nice feel. In the middle ages they used coin-like Jetons. Pennies work well.)

2nd abacus design (each hand has 5 digits, 2 make ten):

```
------------- 10000s
              5000s
------------- 1000s
              500s
------------- 100s
              50s
------------- 10s
              5s
X------------ 1s
```

9834 with 16 pebbles:

```
------------- 10000s
  0           5000s
--0000------- 1000s
  0           500s
--000-------- 100s
              50s
--000-------- 10s
              5s
X-0000------- 1s
```

3rd abacus design:
(opposites, yin-yang, male-female, left-right):

```
  -     +
-----|----- 10000s
     |      5000s
-----|----- 1000s
     |      500s
-----|----- 100s
     |      50s
-----|----- 10s
     |      5s
-----X----- 1s
```

Noting that 9=10-1=IX, 8=10-2=IIX, 4=5-1=IV, and 3=5-2, 9834 uses only 10 pebbles:

```
  -     +
-----|0---- 10000s
     |      5000s
----0|-0--- 1000s
     |      500s
---00|----- 100s
     |0     50s
---00|----- 10s
     |0     5s
----0X----- 1s
```

But since 100=50+50 and 10=5+5 (demotion):

```
  -     +
-----|0---- 10000s
     |      5000s
----0|-0--- 1000s
     |      500s
----0|----- 100s
   00|0     50s
----0|----- 10s
   00|0     5s
----0X----- 1s
```

and since -k+k=0 (cancellation):

```
  -     +
-----|0---- 10000s
     |      5000s
-----|----- 1000s
     |      500s
----0|----- 100s
    0|      50s
----0|----- 10s
    0|      5s
----0X----- 1s
```

So the value of 9834 can be represented with 6 pebbles!
(9834 = 10000 - 166)

=====

9,834,000,000,000,000 = 0.9834 x 10^16

4th abacus design:

```
  -     +
-----|----- 100
     |      50
-----|----- 10
     |      5
-----X----- 1
  Exponent

  Fraction
-----X----- 1
     |      5/10
-----|----- 1/10
     |      5/100
-----|----- 1/100
     |      5/1000
-----|----- 1/1000
     |      5/10000
-----|----- 1/10000
```

Only 9 pebbles are needed to represent 9,834,000,000,000,000:

```
  -     +
-----|----- 100
     |      50
-----|-0--- 10
     | 0    5
-----X-0--- 1
  Exponent

  Fraction
-----X-0--- 1
     |      5/10
-----|----- 1/10
     |      5/100
----0|----- 1/100
    0|      5/1000
----0|----- 1/1000
    0|      5/10000
----0|----- 1/10000
```

=====

9,834 - 4,983 + 3,894 . . .

9,834 - 4,983 = (0.9834 + (-0.4983)) x 10^4:

```
  -     +
-----|----- 100
     |      50
-----|----- 10
     |    0 5
0----X----- 1
  Exponent

  Fraction
-----X----0 1
    0|      5/10
---0-|0---- 1/10
     |      5/100
0--0-|0---- 1/100
0    |      5/1000
0----|00--- 1/1000
0   0|      5/10000
0----|00--- 1/10000
```

Note how the pebbles for 0.9834 are slid away from the vertical bar to make room for the pebbles of the next addend. Then reading -0.4983 naturally from left to right, each digit's pebbles are entered next to the vertical bar from the top. Then you can read them again to check for entry errors before combining with previous results (built-in error checking). You can't check entries before combining on constrained bead abaci like the Soroban.

```
9,834 - 4,983 + 3,894
 = ((0.9834 + (-0.4983)) + 0.3894) x 10^4:

  -     +
-----|----- 100
     |      50
-----|----- 10
     |    0 5
0----X----- 1
  Exponent

  Fraction
-----X----- 1
     |0   0 5/10
---00|-0--- 1/10
     |      5/100
0--00|-0--- 1/100
0    |      5/1000
----0|----- 1/1000
     |0     5/10000
----0|----0 1/10000
```

Using operations of promotion, canceling opposites, demotion, and adding opposite pairs for readability, results in:

```
  -     +
-----|----- 100
     |      50
-----|----- 10
     |    0 5
0----X----- 1
  Exponent

  Fraction
-----X-0--- 1
     |      5/10
---00|----- 1/10
     | 0    5/100
-----|00--- 1/100
     |0     5/1000
----0|----- 1/1000
     |0     5/10000
-----|----- 1/10000
```

Which is read directly as 0.8745 x 10^4 = 8,745.

The number of pebbles needed to do the addition calculation is 18, including the two for the exponent. If the 3rd abacus design is used instead, only 16 pebbles are needed.

It took me (a rank beginner abacist) roughly 2.5 minutes to do the calculation on a 3rd design abacus versus 1/3 of a minute to do it on a modern digital calculator, or by hand. But when the abacus was used heavily by our ancestors they did not have any mechanical calculators and their numerals did not lend themselves well to calculations by hand. Indeed, go back early enough and they did not have paper. How long would it take you to do the calculation using a reed stylus to make marks on wet clay tablets, esp. if you had to cut and sharpen the reed styluses and dig and mix the clay, then form it into tablets?

=====

The structure of the 4th abacus design is that of The Salamis Tablet, the oldest extant artifact of a counting board abacus (http://www.ee.ryerson.ca/~elf/abacus/history.html).

The Salamis Tablet was used with base 10 numbers by the Ancient Greeks as early as 300 BC. But they borrowed the design from the Ancient Babylonians and Sumerians who used it and its precursors (with base 60 sexagesimal numbers) as early as 2700 BC. (http://en.wikipedia.org/wiki/Sumer)

There are indications that the Ancient Romans used the structure of The Salamis Tablet for their base 10 integers and base 12 fractions, and that the Ancient Egyptians used it too. (http://preview.tinyurl.com/EgyptianAbacus).

To see demonstrations of the use of a counting board abacus, including multiplication and division, watch these videos: http://www.youtube.com/user/sks23cu#p/c/0/4Qp5JcceUEM.

Conjecture: The Ancient Babylonians wrote their base 60 numbers in place value notation, just like we write our base 10 numbers, but without indicating any radix point. Their system has been criticized as primitive because of this. However, if you knew that all your numbers were fractions of a whole, where the whole is determined from the context that's always provided, why would you need a radix point? Or trailing zeros, for that matter?

Puzzle this: Ancient Babylonian sexagesimal fractions contained 1-5 digits. What's the probability of one or more embedded zero digits? Without a zero symbol, what would a scribe do with 104?

===== Answer to “Puzzle this” =====

Number of 1 digit numbers = 59
Number of 2 digit numbers = 59 * 59
Number of 3 digit numbers = 59 * 60 * 59
Number of 4 digit numbers = 59 * 60 * 60 * 59
Number of 5 digit numbers = 59 * 60 * 60 * 60 * 59

Total number of numbers
= 59 * (1 + 59 * (1 + 60 + 60^2 + 60^3)

Review:
60 * S - S = (60 + 60^2 + 60^3 + 60^4) - (1 + 60 + 60^2 + 60^3)
S = (60^4 - 1) / (60 - 1)

Total number of numbers
= 59 * (1 + 59 * (60^4 - 1) / 59)
= 59 * 60^4

Total number of numbers without embedded zeros
= 59 * (1 + 59 + 59^2 + 59^3 + 59^4)
= 59 * (59^5 - 1) / (59 - 1)
= 59 * (59^5 - 1) / 58

P(a number w/o embedded zeros)
= (59 * (59^5 - 1) / 58) / (59 * 60^4)
= (59^5 - 1) / (58 * 60^4)

P(a number with embedded zeros)
= 1 - (59^5 - 1) / (58 * 60^4)
~= 0.048898070987654 ~= 5%, or 1 in 20.

Since a zero symbol was not part of the numerals of Ancient Babylonia, when one or more occurred in an abacus result a scribe could do a couple of things:
1. express the result as a sum of numbers with different exponents; or
2. change the units of the result.

An example in decimal numbers:
104 yards = 1 hundred yards and 4 yards;
or, 104 yards = 312 feet.

N.B.: The Ancient Romans, Greeks, and Egyptians didn’t have a zero symbol either, but their numerals contained complete information to place pebbles on a counting board abacus in the right position. E.g., 2009 = MMIX in Roman Numerals.

To the Classroom Teacher (K-18):

The answers to the Pebble Puzzle show a logical sequence of counting board abaci that can represent numbers ranging from whole numbers through positive and negative integers to rational numbers expressed in close-to scientific notation.

These abaci are extremely cost effective; just draw lines on a piece of paper and use aquarium pebbles (or pennies).

The things you can do with these abaci that you can't do with constrained bead abaci like the Soroban are:
1. error check an entry before accumulating (more forgiving for the young and/or novice);
2. represent positive and negative numbers on the same device;
3. represent close-to scientific notation (for older students);
4. perform all four arithmetic operations, including multiplication as repeated addition and division as repeated subtraction, without memorizing multiplication tables; and
5. do all the above in base 10 decimal (Roman, Greek, Egyptian), base 12 duodecimal (Roman fractions), or base 60 sexagesimal (Sumerian & Babylonian) numbers (advanced High School students, or Undergraduate or Graduate College students taking a History of Math or History of Computing course).

My fee for use of these counting board abaci designs in the classroom: mention me as the author, call the 4th design abacus above by the name "The Stephenson Abacus", and send me an email telling me how you're using the abaci.

Hope you find this helpful,
-Steve Stephenson
Math Teacher, Lowell High School, MA; http://sks23cu.net/MT/

P.S.: My original interest in abaci started in Tokyo in 1964 when I bought a Soroban with the book "The Japanese Abacus: Its Use and Theory", Kojima, Takashi, 1954. It sparked my interest in computers. After getting my M.E.E. at Cornell, I worked in EE and the computer field for 30 years. When I decided to become a high school math teacher at the end of 2000, I took a History of Math course as part of my M.Ed. at Univ. of Mass. Lowell. I was struck by how easy it would be to use ancient Roman, Greek, Egyptian, and Babylonian numerals to record abaci calculation results. Prof. Gonzalez said, "Yes, but how would you do multiplication and division?" So I figured out how to use The Salamis Tablet for all four arithmetic operations with numbers expressed in ancient Roman, Greek, Egyptian, or Babylonian numerals. And my methods work ... very well!